Pearson Edexcel

Mark Scheme (Result)

November 2021

Pearson Edexcel GCE Further Mathematics
Advanced Level in Further Mathematics
Paper 9FM0/4A

Edexcel and BTEC Qualifications

Edexcel and BTEC qualifications are awarded by Pearson, the UK's largest awarding body. We provide a wide range of qualifications including academic, vocational, occupational and specific programmes for employers. For further information visit our qualifications websites at www.edexcel.com or www.btec.co.uk. Alternatively, you can get in touch with us using the details on our contact us page at www.edexcel.com/contactus.

Pearson: helping people progress, everywhere

Pearson aspires to be the world's leading learning company. Our aim is to help everyone progress in their lives through education. We believe in every kind of learning, for all kinds of people, wherever they are in the world. We've been involved in education for over 150 years, and by working across 70 countries, in 100 languages, we have built an international reputation for our commitment to high standards and raising achievement through innovation in education. Find out more about how we can help you and your students at: www.pearson.com/uk

November 2021
Publications Code 9FM0_4A_2111_MS
All the material in this publication is copyright
© Pearson Education Ltd 2021

General Marking Guidance

- All candidates must receive the same treatment. Examiners must mark the first candidate in exactly the same way as they mark the last.
- Mark schemes should be applied positively. Candidates must be rewarded for what they have shown they can do rather than penalised for omissions.
- Examiners should mark according to the mark scheme not according to their perception of where the grade boundaries may lie.
- There is no ceiling on achievement. All marks on the mark scheme should be used appropriately.
- All the marks on the mark scheme are designed to be awarded. Examiners should always award full marks if deserved, i.e. if the answer matches the mark scheme. Examiners should also be prepared to award zero marks if the candidate's response is not worthy of credit according to the mark scheme.
- Where some judgement is required, mark schemes will provide the principles by which marks will be awarded and exemplification may be limited.
- When examiners are in doubt regarding the application of the mark scheme to a candidate's response, the team leader must be consulted.
- Crossed out work should be marked UNLESS the candidate has replaced it with an alternative response.

General Instructions for Marking

1. The total number of marks for the paper is 75.

2. The Edexcel Mathematics mark schemes use the following types of marks:

- M marks: method marks are awarded for 'knowing a method and attempting to apply it', unless otherwise indicated.
- A marks: Accuracy marks can only be awarded if the relevant method (M) marks have been earned.
- B marks are unconditional accuracy marks (independent of M marks)
- Marks should not be subdivided.

3. Abbreviations

These are some of the traditional marking abbreviations that will appear in the mark schemes.

- bod – benefit of doubt
- ft – follow through
- the symbol √ will be used for correct ft
- cao – correct answer only
- cso - correct solution only. There must be no errors in this part of the question to obtain this mark
- isw – ignore subsequent working
- awrt – answers which round to
- SC: special case
- oe – or equivalent (and appropriate)
- dep – dependent
- indep – independent
- dp decimal places
- sf significant figures
- ∗ The answer is printed on the paper
- ⌈ The second mark is dependent on gaining the first mark

4. All A marks are 'correct answer only' (cao.), unless shown, for example, as A1 ft to indicate that previous wrong working is to be followed through. After a misread however, the subsequent A marks affected are treated as A ft, but manifestly absurd answers should never be awarded A marks.

Question	Scheme	Marks	AOs
1	Identifies either 3 or 11 as a prime divisor of 66 and proceeds to apply the divisibility test for this prime number.	**M1**	3.1a
	Either $2-0+2-1+0-5+2-0=0=0\times11$ hence *n* is divisible by 11 Or $2+0+2+1+0+5+2+0=12=4\times3$ hence *n* is divisible by 3	**A1**	2.2a
	Both $2-0+2-1+0-5+2-0=0=0\times11$ hence *n* is divisible by 11 And $2+0+2+1+0+5+2+0=12=4\times3$ hence *n* is divisible by 3	**A1**	2.2a
	As also *n* is even, it is divisible by 2, and hence as divisible by 2, 3 and 11, is divisible by $2\times3\times11=66$	**A1**	2.4
		(4)	
			(4 marks)

Notes:

M1: Identifies one of the odd prime factors of 66 and proceeds to check divisibility for it.

A1: Correct method and deduction for either divisibility by 3 or by 11

A1: Correct method and deduction for both divisibility by 3 and by 11

A1: Notes also divisibility by 2 and explains why divisibility by 66 follows. The explanation may have been given in a preamble "66 = 2×3×11 so divisible by 66 if divisible by 2, 3 and 11"

The must be a correct reason for divisibility by 2, ie "it is even" or "last digit is even". Do not accept "last digit is 0" with no reason given.

NB There is no divisibility test for 6, so attempting such will result in the loss of the last two A marks. (E.g. sum of digits being a multiple of 6 is an incorrect test, as for instance 15 satisfies 1+5 = 6 but is not a multiple of 6.)

Question	Scheme		Marks	AOs
2(a)	For $m, n \in \mathbb{Z}_0^+$ we have $m-n \in \mathbb{Z}$ (difference of integers is an integer) and so $\lvert m-n\rvert \in \mathbb{Z}_0^+$, hence closed under $\star$.		**B1**	2.4
			(1)	
(b)	For $m \in \mathbb{Z}_0^+$, $0 \star m = \lvert 0-m\rvert = \lvert -m\rvert = m$	Checks either side	**M1**	1.1b
	and $m \star 0 = \lvert m-0\rvert = \lvert m\rvert = m$ Hence 0 is an identity*.	Checks both sides and makes conclusion.	**A1***	2.1
			(2)	
(c)	For $m \in \mathbb{Z}_0^+$, we need $\lvert m-n\rvert = 0 \Rightarrow n = ...$ or shows $\lvert m-m\rvert = \lvert 0\rvert = 0$		**M1**	2.2a
	As $\lvert m-m\rvert = 0$ for all $m \in \mathbb{Z}_0^+$ each m is self-inverse.		**A1**	2.1
			(2)	
(d)	Checks associativity – ie evaluates $m \star (n \star p)$ and $(m \star n) \star p$ with letter or numbers.		**M1**	1.2
	E.g, $1 \star (2 \star 3) = 1 \star \lvert 2-3\rvert = 1 \star 1 = 0$ but $(1 \star 2) \star 3 = \lvert 1-2\rvert \star 3 = 1 \star 3 = \lvert 1-3\rvert = 2$		**M1**	3.1a
	$1 \star (2 \star 3) \neq (1 \star 2) \star 3$ hence not associative so not a group.		**A1**	2.4
			(3)	
			(8 marks)	

Notes:

(a)

B1: Checks difference of two non-negative integers is an integer and hence its modulus is a non-negative integer and concludes closure. "Always positive" as a conclusion is B0 without consideration of the equal zero case.

(b)

M1: Checks that 0 is a left or a right identity.

A1*: Checks 0 works both sides as an identity and makes conclusion it is an identity.

(c)

M1: Realises m must be its own inverse for each m – accept if just stated m is self-inverse with no proof, or if an attempt is made to show it is self-inverse, or for an attempt to solve $\lvert m-n\rvert = 0$

A1: Each element is self-inverse with a **full** proof given.

(d)

M1: Realises associativity must be checked in some way – may be by producing a counter example, or by attempting to evaluate both sides of the associativity axiom for a general case. A statement of the correct identity is sufficient for the mark to be awarded.

M1: Produces a suitable counter example and evaluates both sides of associativity equation. Attempts at algebraic proofs are unlikely to succeed but allow the method for e.g consideration of. $m > n > p$ giving $\lvert\lvert m-n\rvert - p\rvert = \lvert m-n-p\rvert$ and $\lvert m - \lvert n-p\rvert\rvert = \lvert m-n+p\rvert$ but must have a correct reason to disambiguate the inner moduli. If in doubt use review.

A1: Must have provided a counter example. Deduces associativity does not hold and concludes $\mathbb{Z}_0^+$ is not a group under $\star$

Question	Scheme	Marks	AOs
3(a)	$125 = 87\times1+38$ $87 = 38\times2+11...$	**M1**	1.1b
	$38 = 11\times3+5$ $11 = 5\times2+1$	**M1** **A1**	1.1b 1.1b
	$1 = 11-5\times2$ $= 11-(38-11\times3)\times2 = 11\times7-38\times2$ $= (87-38\times2)\times7-38\times2 = 87\times7-38\times16$	**M1**	2.1
	$1 = 87\times7-(125-87\times1)\times16 = -16\times125+23\times87$ (So $a=-16$ and $b=23$)	**A1**	1.1b
		(5)	
(b)	From (a) $23\times87\equiv1(\text{mod}125)$ so multiplicative inverse of 87 is 23.	**B1ft**	2.2a
		(1)	
(c)	$x\equiv23\times16(\text{mod }125)$	**M1**	1.1b
	$x\equiv368\equiv118(\text{mod }125)$	**A1**	1.1b
		(2)	
			(8 marks)

Notes:

(a)
M1: Begins the process of applying the Euclidean algorithm, with attempt at the first two steps. Allow slips.
M1: Completes the process to the stage shown – if errors have been made at least three steps should have been made in reaching their final line (ending +1) to score this mark.
A1: Algorithm correctly carried out – as shown.
M1: Starts the process of back substitution – at least two substitutions made.
A1: Completes the process and finds the correct values for *a* and *b*.

(b)
B1ft: Deduces correct multiplicative inverse. Accept 23 or anything congruent to 23 modulo 125 or follow through their *b*.

(c)
M1: Multiplies 16 by their multiplicative inverse or any other full method to proceed to the solution, e.g. multiplying the identity found in (a) through by 16 and reducing modulo 125.
A1: $x\equiv118(\text{mod }125)$. Accept 118 or anything congruent to 118 modulo 125 as long as it is part of a correct modulo statement. (Do not accept just 368 on its own.)

Question	Scheme	Marks	AOs
4(i)	(Order of a subgroup must divide the order of a group by Lagrange's Theorem), so need to check if 11 (and/or 21) divides $46^{46}+47^{47}$ and by FLT, e.g. $a^{11-1}=a^{10}\equiv 1(\text{mod } 11)$, so	**M1**	1.1b
	$46^{46}+47^{47}\equiv 2^{4\times10+6}+3^{4\times10+7}\equiv 2^6+3^7\equiv 64+(3^3)^2\times 3$ $\equiv 9+5^2\times 3\equiv 84\equiv 7(\text{mod } 11)$	**M1**	3.1a
	Hence 11 is not a divisor of $46^{46}+47^{47}$ so not a possible order for a subgroup.	**A1**	2.2a
(ii)	$21=7\times 3$ so need to check for factors of 7 and 3, using $a^2\equiv 1(\text{mod } 3)$ and $a^6\equiv 1(\text{mod } 7)$	**M1**	3.1a
	$46^{46}+47^{47}\equiv 1^{46}+2^{47}\equiv 1+2^{2\times23+1}\equiv 1+2^1\equiv 3\equiv 0(\text{mod } 3)$	**M1**	1.1b
	$46^{46}+47^{47}\equiv 4^{46}+(-2)^{47}\equiv 4^{6\times7+4}+(-2)^{6\times7+5}\equiv 4^4+(-2)^5$ $\equiv 16^2-32\equiv 9^2-4\equiv 81-4\equiv 77\equiv 0(\text{mod } 7)$	**M1**	2.1
	As $46^{46}+47^{47}$ divisible by both 3 and 7 it is divisible by 21 and hence this is a possible order for a subgroup.	**A1**	2.4
		(7)	
		(7 marks)	

Notes:

(i)

M1: For an attempt to apply a correct Fermat's Little theorem at least once in the question with either $p = 11$, p = 7 or $p = 3$ on either the 46^{46} or 47^{47} term.

M1: Applies FLT and congruence arithmetic fully to find the residue of $46^{46}+47^{47}$ modulo 11. There will be lots of different routes, so look for an attempt to apply FLT that leads to determining if 11 is a divisor or not.

A1: $46^{46}+47^{47}\equiv 7(\text{mod } 11)$ (accept equivalents as long as it is clear it is not congruent to 0) and deduces it is not a possible order for a subgroup.

(ii)

M1: Applies checks for both 7 and 3 as divisors of $46^{46}+47^{47}$ via similar strategy.

M1: Applies FLT with $p = 3$ to find a smaller residue modulo 3. Other routes are possible.

M1: Applies FLT with $p = 7$ to find a smaller residue modulo 7. Other routes are possible.

A1: Shows $46^{46}+47^{47}$ congruent to 0 modulo 3 and modulo7, and deduces 21 divides $46^{46}+47^{47}$ hence it is a possible order for a subgroup.

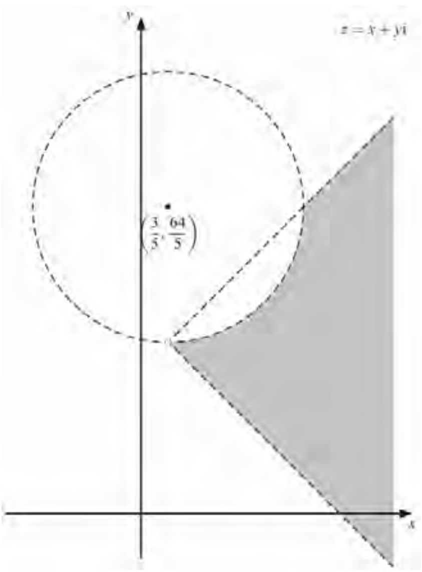
Alt:

M1: Reduces the bases modulo 21 and applies a power reduction technique using congruences for at least one of the power of 46 or 47

M1: Reduces fully by congruence arithmetic either the 46^{46} or 47^{47} term.

M1: Reduces fully by congruence arithmetic both the 46^{46} and 47^{47} terms

A1: Shows $46^{46}+47^{47}$ congruent to 0 modulo 21, and deduces 21 divides $46^{46}+47^{47}$ hence it is a possible order for a subgroup.

Question	Scheme	Marks	AOs
5(a)	$z = x + \mathrm{i}y \Rightarrow \lvert x+9+\mathrm{i}y \rvert = 4\lvert x+(y-12)\mathrm{i} \rvert$	M1	1.1b
	$\Rightarrow (x+9)^2 + y^2 = 16\left(x^2 + (y-12)^2\right)$	M1 A1	1.1b 1.1b
	$\Rightarrow 15x^2 + 15y^2 - 18x - 384y = 81 - 16\times 12^2$ $\Rightarrow x^2 + y^2 - \frac{6}{5}x - \frac{128}{5}y = -\frac{741}{5}$		
	$\Rightarrow \left(x-\frac{3}{5}\right)^2 - \left(\frac{3}{5}\right)^2 + \left(y-\frac{64}{5}\right)^2 - \left(\frac{64}{5}\right)^2 = -\frac{741}{5}$ $\left(\Rightarrow \left(x-\frac{3}{5}\right)^2 + \left(y-\frac{64}{5}\right)^2 = 16\right)$	M1	2.1
	centre $\frac{3}{5} + \frac{64}{5}\mathrm{i}$ **or** radius 4	A1	2.2a
	centre $\frac{3}{5} + \frac{64}{5}\mathrm{i}$ **and** radius 4	A1	2.2a
		(6)	
(b)	 Circle, with centre in correct quadrant for their answer to (a)	M1	1.1b
	Pair of rays at roughly 45° to horizontal, with source in first quadrant OR on the circle.	M1	1.1b
	Correct circle and rays, circle with centre in first quadrant and spanning only quadrant 1 and 2 and pair of rays at roughly 45° to horizontal, meeting at the bottom point of the circle	A1	3.1a
	Region between rays and outside circle shaded	B1ft	3.1a
		(4)	
		(10 marks)	

Notes:

(a)

M1: Applies $z = x + \mathrm{i}y$ to the given equation. Use of other letters, eg $z = u + \mathrm{i}v$ is fine.

M1: Squares and uses modulus to achieve $(x+a)^2 + y^2 = K\left(x^2 + (y+b)^2\right)$

A1: Correct equation, need not be expanded. Award when first seen.

M1: Expands, gathers terms and completes the square.

A1: Either centre or radius correct. Accept coordinates for centre. **A1:** Correct centre and radius. Accept coordinates for centre.
(b) **M1:** Sketches their circle on an Argand diagram. Look for the centre being in the correct quadrant for their answer to (a). **M1:** Pair of rays added to the sketch, at angles $\frac{\pi}{4}$ above and below the horizontal with vertex in the first quadrant OR somewhere on the circle. Need not stem from base of circle for this mark if it stems from the first quadrant, but if not in the first quadrant it must stem from the circle. **A1:** Circle (or arc) in correct position, centre in first quadrant that would span quadrants 1 and 2, with pair of rays at roughly 45° to horizontal, meeting at the bottom point of the circle. **B1ft:** Area outside the circle and between the rays (minor sector) shaded provided the rays span approximately a 90° sector. NB Only the region is asked for, so allow the marks above if only the relevant part of the circle is shown.

Question	Scheme	Marks	AOs
6	$n=1: u_1=(-3)^1\times 1!=-3$ $n=2: u_2=(-3)^2\times 2!=9\times 2=18$ Hence true for n = 1 and n = 2	**B1**	2.2a
	Assume true for some n = k and n = k + 1, so $u_k=(-3)^k k!$ and $u_{k+1}=(-3)^{k+1}(k+1)!$	**M1**	2.4
	Then $u_{k+2}=9(k+1)^2\left((-3)^k k!\right)-3\left((-3)^{k+1}(k+1)!\right)$	**M1**	1.1b
	$=(-3)^k k!\left[9(k+1)^2-3(-3)(k+1)\right]$	**M1**	1.1b
	$=(-3)^k k!\left[9(k+1)(k+1+1)\right]=(-3)^k\times(-3)^2\times(k+1)(k+2)k!$ $=(-3)^{k+2}(k+2)!$	**A1**	2.1
	Hence if true for n = k and n = k +1 then true for n = k + 2. As also true for n = 1 and n = 2, then true for all $n\in\mathbb{N}$ by mathematical induction.	**A1**	2.4
		(6)	
		(6 marks)	

B1: Checks the closed form works for n = 1 and n = 2

M1: Makes the inductive assumption. May use e.g. $n=k-2$ and $n=k-1$ instead and show true for $n=k$. It must be clear it is the closed forms they are assuming, not a recurrence form.

M1: Substitutes expression for $n=k$ and $n=k+1$ (or equivalents) into the recurrence formula.

M1: Takes out common factors of at least $(-3)^k k!$ in their expression, or equivalent for their assumed true values. Treatment of the (-3) must be correct, but condone invisible brackets if recovered.

Note: they may well take out more at this stage, which is fine, e.g.

$$u_{k+2}=9(k+1)^2\left((-3)^k k!\right)-3\left((-3)^{k+1}(k+1)!\right)=(-3)^{k+2}(k+1)!\left[(k+1)+1\right]$$

A1: Simplifies correctly to the required form for their assumed true values.

A1: Correct conclusion made. Depends on all three M's and the A being gained. Must convey the ideas of 1) true for n = 1 and n =2 , 2) if true for two successive cases, it is also true for the next case and 3) a suitable conclusion that it is true for all positive n.

Question	Scheme	Marks	AOs
7(a)	$I_n = \int t^{n-1} \times t\sqrt{4+5t^2}\ \mathrm{d}t = t^{n-1} \times K\left(4+5t^2\right)^{\frac{3}{2}} - \int (n-1)t^{n-2} \times K\left(4+5t^2\right)^{\frac{3}{2}}\ \mathrm{d}t$	M1	3.1a
	$I_n = t^{n-1} \times \frac{2}{3\times 10}\left(4+5t^2\right)^{\frac{3}{2}} - \int (n-1)t^{n-2} \times \frac{2}{3\times 10}\left(4+5t^2\right)^{\frac{3}{2}}\ \mathrm{d}t$	A1	1.1b
	$= t^{n-1} \times \frac{1}{15}\left(4+5t^2\right)^{\frac{3}{2}} - \frac{(n-1)}{15}\int t^{n-2}\left(4+5t^2\right)^{\frac{1}{2}} \times \left(4+5t^2\right)\mathrm{d}t$ $= \frac{t^{n-1}}{15}\left(4+5t^2\right)^{\frac{3}{2}} - \frac{4(n-1)}{15}\int t^{n-2}\left(4+5t^2\right)^{\frac{1}{2}}\ \mathrm{d}t - \frac{5(n-1)}{15}\int t^n\left(4+5t\right)^{\frac{1}{2}}\ \mathrm{d}t$	M1	3.1a
	$\Rightarrow 15I_n = t^{n-1}\left(4+5t^2\right)^{\frac{3}{2}} - 4(n-1)I_{n-2} - 5(n-1)I_n \Rightarrow I_n = \ldots$	M1	1.1b
	$I_n = \frac{t^{n-1}}{5(n+2)}\left(4+5t^2\right)^{\frac{3}{2}} - \frac{4(n-1)}{5(n+2)}I_{n-2}$ *	A1*	2.1
		(5)	
(b)	Surface area $= 2\pi\int_0^1 y\sqrt{\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2}\ \mathrm{d}t$	B1	1.1a
	$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{5}{\sqrt{5}}t^4$ and $\frac{\mathrm{d}y}{\mathrm{d}t} = 2t^3$	B1	1.1b
	$\int y\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\ \mathrm{d}t = \int \frac{1}{2}t^4\sqrt{\left(\frac{5}{\sqrt{5}}t^4\right)^2 + \left(2t^3\right)^2}\ \mathrm{d}t$	M1	1.1b
	$= \int \frac{1}{2}t^4\sqrt{5t^8 + 4t^6}\ \mathrm{d}t = \frac{1}{2}\int t^7\sqrt{4+5t^2}\ \mathrm{d}t$	M1	2.1
	Hence surface area $= \pi\int_0^1 t^7\sqrt{4+5t^2}\ \mathrm{d}t$ *	A1*	1.1b
		(5)	
(c)	$[I_1]_0^1 = \left[\frac{1}{15}\left(4+5t^2\right)^{\frac{3}{2}}\right]_0^1 = \frac{27}{15} - \frac{8}{15} = \frac{19}{15}\quad (=1.266\ldots)$	B1	2.2a
	$\int_0^1 t^7\sqrt{4+5t^2}\ \mathrm{d}t = \left[\frac{t^6}{5\times 9}\left(4+5t^2\right)^{\frac{3}{2}}\right]_0^1 - \frac{4\times 6}{5\times 9}[I_5]_0^1$	M1	1.1b
	$= \frac{3}{5} - \frac{8}{15}\left(\left[\frac{t^4}{5\times 7}\left(4+5t^2\right)^{\frac{3}{2}}\right]_0^1 - \frac{4\times 4}{5\times 7}[I_3]_0^1\right)$ $= \frac{3}{5} - \frac{8}{15}\left(\frac{27}{35} - \frac{16}{35}\left(\left[\frac{t^2}{5\times 5}\left(4+5t^2\right)^{\frac{3}{2}}\right]_0^1 - \frac{4\times 2}{5\times 5}[I_1]_0^1\right)\right)$	M1	3.1a
	Total surface area is $(\pi)\left[\frac{3}{5} - \frac{8}{15}\left(\frac{27}{35} - \frac{16}{35}\left(\frac{27}{25} - \frac{8}{25}\times\frac{19}{15}\right)\right)\right] = \ldots$	M1	2.1

	=awrt 1.11 (3sf) $\left(=\dfrac{69509\pi}{196875}\right)$	**A1**	1.1b
		(5)	
	For the three method marks if the process is worked the other way: $[I_3]_0^1 = \left[\dfrac{t^2}{5\times5}(4+5t^2)^{\frac{3}{2}}\right]_0^1 - \dfrac{4\times2}{5\times5}[I_1]_0^1 \left(=\dfrac{27}{25}-\dfrac{8\times19}{25\times35}=\dfrac{253}{375}=0.6746...\right)$	**M1**	1.1b
	$[I_5]_0^1 = \left[\dfrac{t^4}{5\times7}(4+5t^2)^{\frac{3}{2}}\right]_0^1 - \dfrac{4\times4}{5\times7}[I_3]_0^1 \left(=\dfrac{27}{35}-\dfrac{16}{35}\times\dfrac{253}{375}=\dfrac{6077}{13125}=0.4630...\right)$ $[I_7]_0^1 = \left[\dfrac{t^6}{5\times9}(4+5t^2)^{\frac{3}{2}}\right]_0^1 - \dfrac{4\times6}{5\times9}[I_5]_0^1 = ...$	**M1**	3.1a
	$=\dfrac{27}{45}-\dfrac{24}{45}\left(\dfrac{6077}{13125}\right)=...$	**M1**	2.1
	=awrt 1.11	**A1**	1.1b
		(15 marks)	

Notes:

(a)
M1: Splits the integrand correctly and applies integration by parts in the correct direction to achieve a form as shown in the scheme.
A1: Correct result of applying parts, need not be simplified.
M1: Splits the integrand to identify I_n and I_{n-2} (or allow if I_{n-1} appears due to error for this mark) in the equation.
M1: Rearranges to make I_n the subject from an equation in I_n and I_{n-2}
A1*: Correct completion to the given result.

(b)
B1: Correct parametric formula for surface area given. Must include the 2π and limits, but these may be added at a later stage and must the 2π must be seen before cancelling occurs.
B1: Correct derivatives of x and y with respect to t seen or implied.
M1: Applies their derivatives and y to $\int y\sqrt{\left(\dfrac{dx}{dt}\right)^2+\left(\dfrac{dy}{dt}\right)^2}\,dt$. May have included the limits and 2π here, but they are not needed for this mark.
M1: Squares the derivatives and takes a common factor t^3 from the square root to reach appropriate form for the integral. Limits and 2π not needed for this mark.
A1*: Reaches correct answer with no errors seen, limits included (but do not need to be justified) and the dt must be present and the 2π must have been seen and correctly processed.

(c)
B1: Correct value for I_1 between the limits - need not be simplified and may be seen later in the working.
M1: Applies the reduction formula from (a) in attempt to solve the integral. This may be from I_7 to I_5 or from I_1 **to** I_3 depending on the direction they are going. Allow for any application relevant to the integral (e.g between two odd values for n).

M1: Applies the reduction formula two more times to link I_1 and I_7. May have evaluated at each stage or find expression before substituting limits but look for the complete process to link the two intervals.

dM1: Applies the limits to their integral in a complete process to reach an answer. Allow if substitution happens throughout the process of reduction or at the end but it must be a complete process to find reach a value, though allow if the π is not included.

A1: Must have scored all three method marks. Correct answer, awrt 1.11.

Question	Scheme	Marks	AOs
8(a)(i)	$\begin{pmatrix}5 & -2 & 5\\0 & 3 & p\\-6 & 6 & -4\end{pmatrix}\begin{pmatrix}2\\1\\-2\end{pmatrix}=\begin{pmatrix}-2\\3-2p\\2\end{pmatrix}=-1\times\begin{pmatrix}2\\2p-3\\-2\end{pmatrix}$		
	Corresponding eigenvalue is -1	B1	1.1b
		(1)	
(a)(ii)	$2p-3=1\Rightarrow p=...$	M1	1.1b
	$p=2$ *	A1*	1.1b
		(2)	
(a)(iii)	$\det\begin{pmatrix}5-\lambda & -2 & 5\\0 & 3-\lambda & p\\-6 & 6 & -4-\lambda\end{pmatrix}=0$ $\Rightarrow(5-\lambda)((3-\lambda)(-4-\lambda)-12)-(-2)(12)+5(6(3-\lambda))=0$	M1	1.1b
	$\Rightarrow\lambda^3-4\lambda^2+\lambda+6=0$	A1	1.1b
	$(\Rightarrow(\lambda+1)(\lambda^2-5\lambda+6)=0\Rightarrow(\lambda+1)(\lambda-2)(\lambda-3)=0)$ Eigenvalues are (-1), 2 and 3	A1	1.1b
	Either $\left.\begin{matrix}3x-2y+5z=0\\y+2z=0\\-6x+6y-6z=0\end{matrix}\right\}$ or $\left.\begin{matrix}2x-2y+5z=0\\2z=0\\-6x+6y-7z=0\end{matrix}\right\}\Rightarrow x/y/z=...$	M1	2.1
	Either $k\begin{pmatrix}3\\2\\-1\end{pmatrix}$ (for $\lambda=2$) or $m\begin{pmatrix}1\\1\\0\end{pmatrix}$ (for $\lambda=3$)	A1	1.1b
	Both $\left.\begin{matrix}3x-2y+5z=0\\y+2z=0\\-6x+6y-6z=0\end{matrix}\right\}$ and $\left.\begin{matrix}2x-2y+5z=0\\2z=0\\-6x+6y-7z=0\end{matrix}\right\}\Rightarrow x/y/z=...$	M1	2.1
	Both $k\begin{pmatrix}3\\2\\-1\end{pmatrix}$ (for $\lambda=2$) and $m\begin{pmatrix}1\\1\\0\end{pmatrix}$ (for $\lambda=3$)	A1	1.1b
		(7)	
(b)	E.g. $\mathbf{P}=\begin{pmatrix}2 & 3 & 1\\1 & 2 & 1\\-2 & -1 & 0\end{pmatrix}$ and $\mathbf{D}=\begin{pmatrix}-1 & 0 & 0\\0 & 2 & 0\\0 & 0 & 3\end{pmatrix}$	B1ft	2.2a
		(1)	
(c)(i)	$\dot{u}=ku\Rightarrow\int\frac{1}{u}\mathrm{d}u=k\int\mathrm{d}t\Rightarrow\ln u=kt(+c)$	M1	1.1b

	So $u = Ae^{kt}$ or $u = e^{kt+c}$	**A1**	1.1b
		(2)	
(ii)	$\begin{pmatrix}\dot{x}\\ \dot{y}\\ \dot{z}\end{pmatrix} = \mathbf{PDP}^{-1}\begin{pmatrix}x\\ y\\ z\end{pmatrix} \Rightarrow \begin{pmatrix}\dot{u}\\ \dot{v}\\ \dot{w}\end{pmatrix} = \mathbf{P}^{-1}\begin{pmatrix}\dot{x}\\ \dot{y}\\ \dot{z}\end{pmatrix} = \mathbf{D}\begin{pmatrix}u\\ v\\ w\end{pmatrix} = \begin{pmatrix}-u\\ 2v\\ 3w\end{pmatrix}$	**M1**	3.1b
	$\Rightarrow \begin{pmatrix}u\\ v\\ w\end{pmatrix} = \begin{pmatrix}Ae^{-t}\\ Be^{2t}\\ Ce^{3t}\end{pmatrix}$	**M1**	2.2a
	$\Rightarrow \begin{pmatrix}x\\ y\\ z\end{pmatrix} = \mathbf{P}\begin{pmatrix}Ae^{-t}\\ Be^{2t}\\ Ce^{3t}\end{pmatrix} = \ldots$	**M1**	3.4
	$\Rightarrow \begin{pmatrix}x\\ y\\ z\end{pmatrix} = \begin{pmatrix}2Ae^{-t} + 3Be^{2t} + Ce^{3t}\\ Ae^{-t} + 2Be^{2t} + Ce^{3t}\\ -2Ae^{-t} - Be^{2t}\end{pmatrix}$	**A1**	1.1b
		(4)	
			(17 marks)

Notes:

(a)(i)
B1: For the correct eigenvalue of -1
(ii)
M1: Correct equation with their eigenvalue set up – need only see middle equation for this.
A1*: Correct proof (full matrix calculation not necessary).
(iii)
M1: Applies $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$ to achieve a cubic in λ (or other variable, simplification not required). Allow with p used instead of 2, and look for two correct "terms" in the expansion leading to a cubic as evidence of the expansion.
A1: Correct simplified cubic. Note this may be implied by correct answers from a calculator following a correct expansion seen for the M.
A1: Correct eigenvalues
M1: Forms and solves eigenvector equations for at least one (other than -1) eigenvalue.
A1: One correct (other) eigenvector
M1: Both eigenvectors attempted.
A1: Both (other) eigenvectors correct.

(b)
B1ft: A correct corresponding **P** and **D**, follow through on their answer to (a). Columns may be in different order, but should be consistent for their **P** and **D**.

(c)(i)
M1: Separates variables and attempts the integration (constant not required).
A1: Correct answer for $u = \ldots$, either form, including constant of integration

(ii)

NB different orderings of the columns of P and D will give the terms in different orders here.

M1: Uses their **P** and **D** to transform system into equation in *u*, *v* and *w* (may be implied).

M1: Forms the solution for *u*, *v* and *w* using their eigenvalues.

M1: Reverses the substitution (multiplies by their **P**) to get solution for *x*, *y* and *z*.

A1: Correct answer, in matrix form or as separate equations – award when first seen and isw.